Axiomatic Logics for ATIS

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Background

Elizabeth Steiner, in her book entitled *Methodology of Theory Building* (Educology Research Associates, Sydney, Australia, 1988), states the following:

> There are four sets of methods involved in building theory. These methods are the two of criticism: explication and evaluation, and the two of construction: emendation and extension.

> While it now is patent that criticism must precede construction, it is not yet obvious that there are steps prior to criticism; one must be able to recognize theory if one is to critique it.

While Steiner provides a guide for determining whether or not statements can be construed as 'theory', it is in the form of interrogatories rather than definitive. Our concern here is with the definitive identification of 'theory' in education. Further, we are concerned with identifying and selecting a certain type of theory—one that is comprehensive, consistent and complete, and can be developed as an axiomatic theory. Steiner asserts:

> The criteria for choice [of a theory] are functionality [that is, syntactic and semantic adequacy] and comprehensiveness (p. 88).

Syntactic adequacy is determined by the logical consistency and decision procedures of the theory; and semantic adequacy is determined by the completeness and proper construction of the theory. Steiner then concludes:

> Constructive moves in theory building, therefore, are either those of emendation or extension (p. 90).

The approach that will be taken in this report is that of emendating existing theory, and to recognize that retroduction is a process of theory building that is an emending of existing theory; whether that is a single theory or multiple theories—that is, an emendation from the whole cloth of relevant knowledge. In order to do so, we must see what the current state is of theory in education and the relevant efforts that have preceded the development of *ATIS*.

As theory building is either that of emendation or extension, it is seen that the retroductive process is one of emendation of existing theory.

In 1950, Ludwig von Bertalanffy wrote "An Outline of General Systems Theory," (*British Journal for the Philosophy of Science*, Vol. 1, No. 2, August 1950). The writing of this report followed his initial attempt at presenting his concepts on General Systems Theory at a lecture in 1937 at the University of Chicago. His initial efforts were not well received, but his report of 1950 has had widespread acceptance and has produced numerous additional related studies.

At about the same time that Bertalanffy was developing his work on General Systems Theory; in 1936 Kurt Lewin was developing a formal theory of the social sciences, his "topological field theory." In this theory, Lewin introduced mathematical terminology to study human behavior. The problem with Lewin's work, however, was that it did not utilize the power of mathematical topology. Although it attempted to present the image of a mathematical theory, in fact it was not, it was a descriptive theory that utilized mathematical concepts but not the mathematics.

For example, Lewin introduced the "mathematical equation" B = f(P,E) with the intention of asserting that 'behavior' is equal to a 'function' of two variables, 'person' and 'environment'. Further, this "function" was supposed to indicate interdependence between 'person' and 'environment', which it does not do. Unfortunately, he could have represented the same or more by simply saying that behavior is determined by, or dependent upon, the individual and the individual's environment, and they are interrelated in a manner that they produce mutual affect relations. The problem is, he gave no mathematical structure or rigor to the purported function. There was no logic or mathematical structure in which the equation or function could be interpreted.

However, Lewin was attempting to give mathematical rigor to an area that Bertalanffy was also developing—the recognition that behaviors, individuals, and environments are all interrelated; that is, they are part of some 'system'.

One problem that both Bertalanffy and Lewin had to confront was the prevailing methodology of classical science. Classical science was dependent on the following techniques for the development of theory: observation, hypothesis, and experiment. This was an inductive process, and one that was counter to what Peirce had already clearly analyzed. Both prior to and after Peirce, the classical development of theory in the social sciences was that of induction. A parallel construction of theory in physics will be presented below. It is instructive, therefore, to recognize that a physicist may consider the development of theory to be that of the classical science, even though it is not. Since a physicist is not so much concerned with the process of theory development as with the development of theory, this confusion is understandable. So, while a physicist may assert that the development of theory is by induction, in fact many physicists proceed in a manner defined by Peirce as retroduction in the vertical development of new theory, or by extension in the horizontal development of existing theory.

Karl Popper recognized the problems with the classical approach to the development of theory, although he continued to ignore Peirce. As an alternative, Popper proposed a new scientific methodology. In his two books, *The Logic of Scientific Discovery* (1968) and *Conjectures and Refutations* (1972), he introduced an alternative to inductive inference for theory building—the hypothetico-deductive scientific method for theory development.

While this approach may appear to be better than the inductive method, it falls short of clearly defining a methodology that will result in scientific theory. In fact, it but jumps to the hypothesis and explicates the "theory" from there. The problem is that no theory has actually been developed.

One problem with the hypothetico-deductive approach to theory development is that there is an assumption that the "hypotheses" are somehow part of a fully developed theory, since otherwise it is known that the hypotheses being tested are just some statements created by a researcher for the sole purpose of carrying out an experiment, even if it is deductively obtained from a hypothesis. Deductive inferences are no more reliable than the hypotheses upon which they are founded.

It should be noted here that in physics and the other mathematical sciences, there is an underlying theory upon which all hypotheses rely. That is, the researcher proceeds from an existing theory, whether that is Newtonian Physics, Einstein Physics, or some other theory, and this theory provides the framework in which the scientist works. Then the research continues as a model is developed that starts to predict what the effects should be. Experiments are then conducted to test the effects. The model is refined and other researchers develop competing models. Since both models cannot be right other researchers proceed to determine which model provides the correct interpretation of what is observed. New predictions are then made and tested.

After much iteration, someone determines how to start at the atomic or subatomic level and develop a theory that is based on certain fundamental axioms of science and ends up predicting exactly what happens. Then, the value of the theory is determined by the predictions that can be made. The predictions of the theory provide new outcomes that no intuition or hypothesis could have predicted. These predictions are a result of the equations that were developed from the theory and relied on no preconceived notions that the effect could even exist.

Possibly the best example of this theory development and results comes from quantum mechanics that has predicted so many counterintuitive events. One example from physics is the *Josephson Effect* in superconductivity. When two superconductors are held close to each other, there is a coupling of the quantum mechanical wave functions between the two superconductors. This coupling was predicted from the equations governing the theory of superconductivity and was quickly confirmed in the laboratory. The *Josephson Effect* has become a valuable tool as a detector of extremely small magnetic fields and in many other practical applications. Here, the theory predicted non-obvious outcomes, the very purpose of a theory.

This sequence of research events is indicative of what is intended for the education researcher and is explicated in this guide for the development of an education axiomatic theory.

The confusion in education is that hypotheses are presented as self-contained assertions, whereas in fact they are not presented in a vacuum. They must be part of some theory structure, or they are nothing more than the opinion of the researcher, even if that opinion is subsequently "validated." It is this process of hypothesis creation that has resulted in numerous "tests" of the same subject area resulting in differing conclusions. The problem is not necessarily the tests, but that the underlying assumptions of the "theory" in which the hypothesis is couched have not been fully recognized. Hypotheses must be generated by theory, they do not create theory nor are they themselves theory.

The problem with this hypothetico-deductive methodology is that it does not produce theory. The theory-building process herein described is an effort to correct that problem by providing a means by which theory can actually be developed. A proper methodology requires testing *theory-derived hypotheses* and all new applications derived from the hypotheses until the evaluations lead to a new theory that describes the problem based on first principles, accepted assumptions.

As an alternative to the hypothetico-deductive methodology, Glaser and Strauss have developed the "Grounded Theory" approach to the development of theory in the social sciences. This approach, wrongly identified as abduction from Peirce, relies on the belief that theory can be inductively discovered as the result of systematically analyzing data. If so, then this approach is very similar to, if not identical to the data mining procedures used to structure unstructured data. The response to each is the same—structuring unstructured data is certainly helpful in recognizing established patterns within systems, but it does not produce theory.

From the work of Steiner, both approaches can be recognized as ill-founded. Steiner resolves the problems by clearly stating the distinctions between retroduction, deduction and induction as presented by Peirce in the 1890's:

Retroduction devises theory.

Deduction explicates theory.

Induction evaluates theory.

An inductive process does not develop theory, whether that induction is used to directly propose a theory, or to develop a hypothesis from which deductive inferences are obtained. Glaser and Strauss recognize the problem with induction and attempt to work around it, however, their attempt to couch their development under the guise of "abduction" fails.

First, as established in Report #1 of this series, abduction is not a retroductive process. They have not carefully read Peirce. They consider abduction as a means of obtaining theory that has been obtained from data patterns—that is, data mining techniques. Theory development is a retroductive process, and not an inductive process. New theory is emended from existing theory, not existing data. In this case, Glaser and Strauss misinterpret the abduction and retroduction processes.

Essentially, the scientific methodology of the social sciences has been hypothesis-driven. That is, whether the theory-building methodology has been defined by induction or hypothetico-deduction, each relies on a hypothesis that is devoid of the foundations required of a legitimate theory.

Legitimate theory is developed by retroduction, whether that retroductive process results from the development of new theory from existing theory or the development of new theory from the *whole cloth* of relevant knowledge. For example, theory can be developed from the existing theories of Set Theory, Information Theory, Graph Theory and General Systems Theory in a very analytic manner, as was done for the development of the SIGGS theory model in 1966 by Maccia (Steiner) and Maccia.[1] Or, theory can be developed from mathematics, education, chemistry, physics and behavioral sciences by recognizing a wholeness of concepts contained in this *whole cloth* that provides a perspective that describes and predicts what is found in education systems.

Further, as Popper and others have recognized, theory must be axiomatic with all of the associated safeguards that such an approach brings. In attempts to do so, the social sciences have claimed to produce theories that have a rigor similar to the physical sciences by abductively introducing mathematical constructs. Logic and mathematics must not simply be brought in, as part of a descriptive theory, but the theory must be developed as a result of the logic and mathematics.

This is where historical and current research in education has failed, as research continues to proceed from a position of validating hypotheses. Education research is hypothesis-driven, rather than theory-driven. While axiomatic logico-mathematical theories are far more difficult and complex than hypothesis-driven methodologies, such theories must be developed if educology is to move beyond a "My Theory" methodology to that of developing a consistent, comprehensive and complete theory of education.

To assist in bridging the gap from hypothesis-driven to axiomatic-theory-driven science, a parallel development in physics will be considered. Even in physics, which is frequently considered as being "proven" or "empirically valid," theories are considered to be acceptable for describing the physical world as a result of a "preponderance of evidence" that they produce accurate predictions of that physical world. The same will hold in the social sciences; that is, a theory is accepted as a result of the "preponderance of evidence" that it produces consistently valid predictions.

Following is an example of the development of a theory in physics.

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[1] Maccia, Elizabeth Steiner & George S. Maccia (Principal Investigators), James F. Andris & Kenneth R. Thompson (Research Assistants), *Development of Educational Theory Derived from Three Educational Theory Models*, Final Report, Project No. 5-0638, Contract #OE4-10-186, U.S. Department of HEW & The Ohio State University, Research Foundation, Columbus, Ohio. (1966)

Rock Theory

Desired Theory: Electrical Properties of Rocks
Existing Theory: Electrical Properties of Glass

It is recognized that the electrical properties of glass may be similar to the electrical properties of rocks. Therefore, the existing Electrical-Glass-Property Theory is used to retroductively develop an Electrical-Rock-Property Theory.

This new theory is an emendation of the existing Glass Theory. As such, it brings with it the basic logic of that theory, which is comparable to other theories in physics.

Levels of Theory Construction

There are several levels of theory construction required, especially in an axiomatic theory, before the actual desired empirical theory is obtained. These levels are discussed below, and an example from physics is presented on the pages following.

The first level consists of defining the *Basic Logics*; that is, the Sentential, Predicate, Class and Relation Calculi.
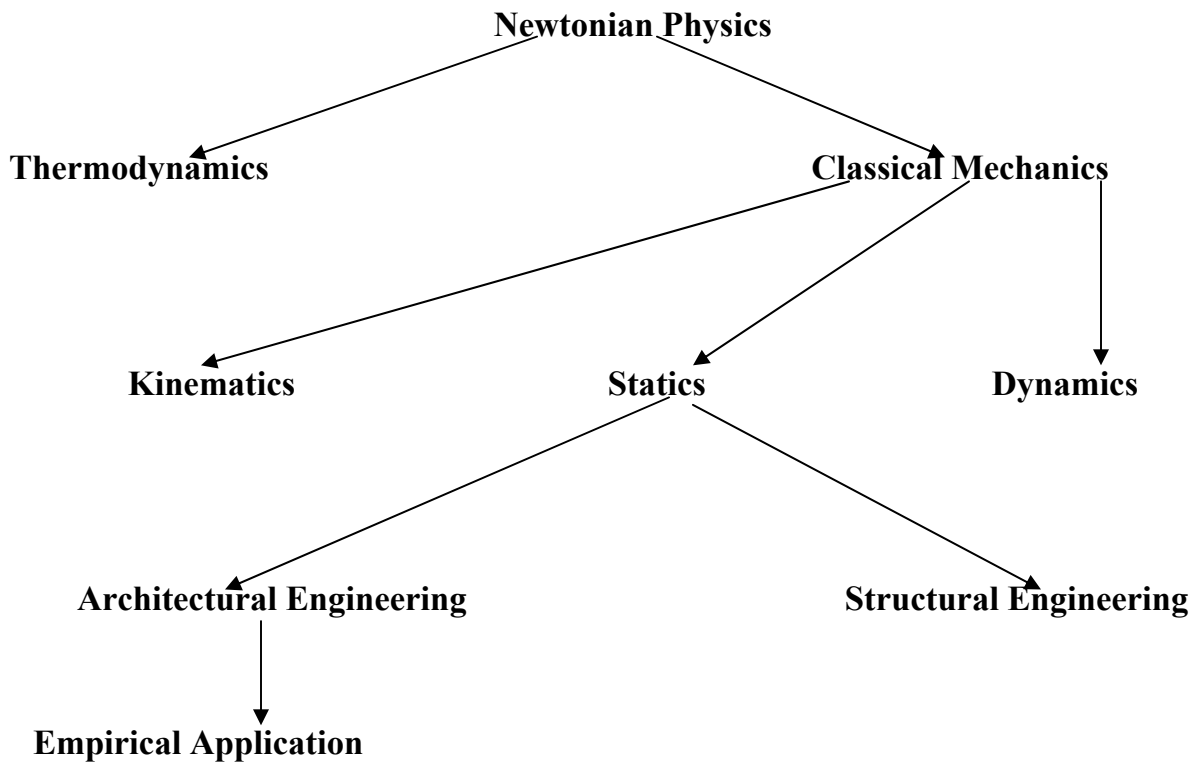
The Basic Logics provide the decisional rules by which theorems are formally derived within the theory. These provide the customary deductive logic used in physics.

Once the decisional logic is determined, then the levels of scientific inquiry must be defined. These will each require its own axioms or other structure that will be used to deduce the outcomes of the theory.

Various areas of physics exemplify the relation between levels of a theory. Then, the axioms or laws will be tracked to exemplify how each higher-order theory affects the lower-order theories. In axiomatic theories, this will be accomplished by the introduction of appropriate axioms at each level.

The following tree diagram depicts various levels of theories in physics in which each lower level is dependent on the axioms or laws of the one above it, but will also introduce axioms, laws or principles that are extended from those above it.

That is, recalling that the development of theory is by emendation or extension, once the initial theory has been designed by an emendation of another theory; for example, by means of a retroductive process, then the theory is constructed by extension. These extensions can be horizontal; that is, within the existing theory, or vertical; that is, by introducing sub-theories. This vertical development is shown in the following diagram. So as not to make the diagram too complex, only one area of physics is extended to the following vertical level below it; for example, "Mechanics" and then "Statics."

Newtonian Physics

Thermodynamics   **Classical Mechanics**

Kinematics   **Statics**   **Dynamics**

Architectural Engineering   **Structural Engineering**

Empirical Application

It is recognized that Thermodynamics has sub-theories extending below it, as do Kinematics and Dynamics. However, we will only consider the development of each of the following theories as one is extended from the one above it:

Newtonian Physics → Classical Mechanics → Statics → Architectural Engineering

Newtonian Physics

At this level we are concerned with the axioms, hypotheses, or principles that are used to explicate Newtonian Physics. In Newtonian Physics these statements are referred to as "laws" or "postulates."

Newton's Laws of Motion:

Newton's first law states that, if a body is at rest or moving at a constant speed in a straight line, it will remain at rest or keep moving in a straight line at constant speed unless a force acts upon it.
This postulate is known as the law of inertia.

Newton's second law states that the time rate of change of the velocity or acceleration, a, is directly proportional to the force F and inversely proportional to the mass m of the body; i.e., $a = F/m$, or $F = ma$.
From the second law, all of the basic equations of dynamics can be derived. As noted previously, this initial theory provides the main assumptions by which all extensions of the theory are obtained.

Newton's third law states that the actions of two bodies upon each other are always equal and directly opposite. The third law is important in statics (bodies at rest) because it permits the separation of complex structures and machines into simple units that can be analyzed individually with the least number of unknown forces.

Newton's law of gravitation is a statement that any particle of matter in the universe attracts any other with a force, F, varying directly as the product of the masses, m_1 and m_2, and a gravitational constant, G, and inversely as the square of the distance between them, R; i.e., $F = G(m_1 m_2)/R^2$.

Thermodynamics   Classical Mechanics

At this level we are concerned with the axioms, hypotheses, or principles that are used to explicate Classical Mechanics.

Classical Mechanics is a theory of the physics of forces acting on bodies.
The first three laws of Newtonian Physics are fundamental to Classical Mechanics and the extensions required for this theory.
In order to consider the problems relevant to Classical Mechanics, the definition of the '*position*' of a '*point particle*' is introduced. This is an extension of Newtonian Physics. With the introduction of a *point particle*, the three laws are used to develop properties relevant to Classical Mechanics. For example, properties relating to force and energy are developed from Newton's Second Law.

Classical Mechanics is subdivided into: Statics, Kinematics, and Dynamics. We will continue our vertical theory development by considering Statics.

Kinematics   Statics   Dynamics

Continued on Next Page

Kinematics   **Statics**   **Dynamics**

At this level we are concerned with the axioms, hypotheses, or principles that are used to explicate Statics.

Statics: Statics is concerned with physical systems that are in static equilibrium. When in static equilibrium, the system is either at rest or moving at constant speed. By Newton's Second Law, this situation implies that the net force and net torque on every subsystem is zero. From this constraint, and the properties developed in Classical Mechanics, such quantities as stress or pressure can be derived.

1. When a wire is pulled tight by a force, F, the **stress**, σ, is defined to be the force per unit area of the wire: $\sigma = F / A$. The amount the wire stretches is called strain.

2. Failure occurs when the load exceeds a critical value for the material; the tensile strength multiplied by the cross-sectional area of the wire, $F_c = \sigma_t A$.

The theory has now been extended to include properties required for the Theory of Statics. From here, specific properties will be required for specific areas of application as shown by the next level.

Although not considered, the following definitions are provided:

Kinematics: Kinematics is the branch of mechanics concerned with the motions of objects without being concerned with the forces that cause the motion.

Dynamics: Dynamics is the branch of mechanics that is concerned with the effects of forces on the motion of objects.

Architectural Engineering   **Structural Engineering**

At this level we are concerned with the axioms, hypotheses, or principles that are used to explicate Architectural Engineering as an application of the Physics Theory of Statics.

For example, at this level the theory extension will be with respect to specific physical problems of concern to architectural engineers. For example, axioms, hypotheses, or principles related to the analysis of architectural structures to preclude structural failures, construction defects, expansive soils, explosions, fires, storms/hail, tornadoes, vehicular impacts and water leaks.

The theory at this level will then be used to analyze specific empirical instances.

Empirical Application

The Argument for a Formal Logic

Elizabeth Steiner, in her book *Methodology of Theory Building*[2], asserts:

> One must understand the many forms (kinds) of theory if one is not to apply the wrong art, i.e., if one is not to criticize or construct theory erroneously.

This same word of caution needs to be applied to the choice of logic that underlies the development of theory. The logic of a theory provides the means by which validity of statements of the theory can be "proved" as "true," and provides the means by which valid statements of the theory are derived. For a scientific theory, normally a *symbolic logic*; that is, *formal logic*, is desired as such provides a means to obtain rigorous proofs for the validity of statements.

The logic required for *ATIS* is that of the *Sentential Calculus* and *Predicate Calculus* that is normally used for mathematics and the mathematical sciences. While both calculi are concerned with analyzing statements based only on the form of the statements, they differ in terms of the types of statements analyzed.

The *Sentential Calculus* is concerned with the form of the aggregate statement with no concern of what is contained within the statement. The *Predicate Calculus* is concerned with the logic of predicates; that is, statements and their constituent parts, as related to quantifiers—normally the universal and existential quantifiers, although others will be required for the logic of *ATIS*.

A *Predicate Calculus* of this type is referred to as a *First Order Predicate Calculus* (FOPC) and is an extension of the *Sentential Calculus*. Whereas the atomic sentences of the *Sentential Calculus* are propositional statements, the atomic sentences of a FOPC are predicates with one or more *arguments*; the number of arguments being the *valence* of the predicate.

Further, whereas there are no quantifiers in the *Sentential Calculus*, FOPC introduces quantifiers over variables.

A FOPC is then extended by a *Second Order Predicate Calculus* (SOPC) which simply introduces quantification over the predicates of the FOPC. Each higher-order calculus but introduces quantification of the sentences of the next-lower order calculus.

For example, a FOPC has statements of the form $\forall x P(x)$; that is, the quantification covers only the variable, x. A SOPC has statements of the form $\forall P(P(x) \supset P(y))$; that is, the quantification covers the predicates. By so doing, a SOPC covers quantification over subsets and relations. The next higher-order calculus would provide quantification for relations of relations. These considerations are a part of Type Theory, which we do not need to address at this time.

The advantage of a symbolic logic is that proofs are dependent only on the form of the statements, and not on their content. The advantage is that while it may take great insight to discover a theorem, once discovered it can be checked very systematically.

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[2] *Methodology of Theory Building*, Elizabeth Steiner, Indiana University, Educology Research Associates, Sydney, 1988.

The emphasis for theory development, however, is that the theoretician must continue to rely on intuition as the primary means of theory development, and the rigors of the basic logic are but a tool to assist in this development. Steiner defines 'intuition' as a "non-inferential form of reasoning. It is a direct intellectual observation of the essence of what is given in experience."[3]

As will be discussed later, the System Logic schemas will be presented in two forms: Those that are derived directly from the axioms and should, therefore, be considered directly descriptive of the system, and those that are "theory construction axioms" and are, therefore, to be evaluated through intuition or available analytic tools before being considered part of the theory. The definition of 'intuition' by Charles Sanders Peirce addresses this desired theory-building method very directly when he states:

> Intuition is the regarding of the abstract in a concrete form, by the realistic hypostatization of relations.[4]

While following well-defined steps can check a schema; a pragmatic logic must guide the development and acceptance of the theory. The need for a pragmatic logic is especially relevant for *ATIS System Construction Theorems* (*SCTs*) that are an integral part of the theory explication. The far-reaching consequences of the introduction of this theory-development methodology is not elsewhere discussed in the literature, as far as this researcher has been able to determine, and will be only referenced herein since there may be important proprietary consequences resulting from its usage. Essentially, the value of such theorem schemas will depend on the rules of construction that are defined for their usage. However, they will be further considered in a later section, to as great a degree as possible, in the section entitled *Significance of SCTs*.

A *formal logic* is dependent on the *formal language* by which the logical concepts are explicated. We will start by defining *formal language*.

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[3] *Methodology*, *Ibid.*, p. 93.

[4] *Collected Papers of Charles Sanders Peirce*, Volume 1, *Principles of Philosophy*, (Editors) Charles Hartshorne and Paul Weiss, The Belknap Press of Harvard University Press, Cambridge (1960), §1.383, p. 203.

Formal Language

A *formal* or *logical language* is a set of rules for constructing *formulas* that can be assigned truth-values. This is the primary advantage of the *ATIS* logical-axiomatic theory—it provides the means to obtain logical validation of its extensions; that is, its empirical applications.

A formal or *logical language*, $\mathscr{L}$, is defined as a list of primitive symbols that are classified in a eleven-tuple:

$\mathscr{L} = (\mathcal{A}, \mathcal{S}, \mathcal{O}, \mathcal{F}, \mathcal{R}, V, \mathrm{G}, \mathcal{T}, \mathcal{Q}, {}_\mathcal{F}V, \mathcal{D})$; where:

- $\mathcal{A}$ is a set of *atomic statements*— p q p_1 q_1 p_2 q_2 ... (where the 3 periods are extra-logical with their common interpretation)
- $\mathcal{S}$ is a set of *statement operators*, one *unary-* and one *binary-atomic statement operators*, respectively, from which the additional statement operators are defined ('$\underline{\vee}$' is the exclusive "or")— $\sim \wedge \quad \vee \underline{\vee} \supset \equiv$
- $\mathcal{O}$ is a set of *atomic objects*— A B A_1 B_1 A_2 B_2 ...
- $\mathcal{F}$ is a set of *object function operators*— $\in \cup \cap \setminus$
- $\mathcal{R}$ is a set of *object relation operators*— $= \subset \equiv$
- V is a set of *atomic variables*— x y x_1 y_1 x_2 y_2 ...
- $\mathcal{G}$ is a set of *formula grouping symbols*, of which there are two, and, by convention, additional grouping symbols are employed— ( ) [ ] { } ⟨ ⟩
- $\mathcal{T}$ is a set of *statement transformation operators*, of which there are two— $\vdash \models$
- $\mathcal{Q}$ is a set of *quantifiers*, of which there is only one *atomic quantifier*, the *universal quantifier*, from which the additional quantifiers are defined ('A' is the *ATIS quantifier*)— $\forall \quad \exists \, \iota \, \hat{w} \, \mathsf{A}$
- ${}_\mathcal{F}V$ is a set of *formula values*— * # d d_1 d_2 ... (where '*' and '#' designate "true" and "false", respectively, and values d, d_1, d_2, ... $\in \mathcal{R}$ where $\mathcal{R}$ is the set of real numbers)
- $\mathcal{D}$ is a set of *value-assignment (descriptor) operators*, of which there is only one— /

From the above categories, we have the following list of *primitive symbols*:

p q p_1 q_1 p_2 q_2 ... $\sim \wedge$ A B A_1 B_1 A_2 B_2 ... $\in \cup \cap \setminus = \subset \equiv$ x y x_1 y_1 x_2 y_2 ... ( ) $\vdash \models \forall$ * # d d_1 d_2 ... /

These primitive symbols are read as follows:

p q p_1 q_1 p_2 q_2 ... are *atomic statements* and are read as denoted.

$\sim \wedge$ are *unary-* and *binary-statement operators*, respectively, and are read "not" and "and," respectively.

A B A_1 B_1 A_2 B_2 ... are *atomic objects* and are read as denoted.

∈ ∪ ∩ \ are *object function operators* and are read "is an element of," "union," "intersection" and "minus," respectively.

= ⊂ ≡ are *object relation operators* and are read "equals," "subset," and "equivalent to," respectively.

א ﺢ א$_1$ ﺢ$_1$ א$_2$ ﺢ$_2$ ... are *atomic variables* and are read as denoted.

( ) are *formula groupings* and are not read.

⊢ is a *transformation operator*, and is read "produces" or "yields."

⊨ is a *transformation operator*, and is read "model of."

∀ is the *universal quantifier*, and is read "for all."

* # ℜ are formula values, and are read "true," "false," and "reals."

/ is a *value-assignment (descriptor) operators*, and is read "is assigned."

While p q p$_1$ q$_1$ p$_2$ q$_2$ ... are atomic statements; in general, a statement is a sequence of primitive symbols. For example, the following is a statement:

$$\wedge p)^* A_1 \cup \sim q_{50}$$

A formula is defined as follows:

(1) Atomic statements are formulas;
(2) If P is a formula, then (~P) is a formula;
(3) If P and Q are formulas, then (P ∧ Q) is a formula;
(4) Only statements are formulas; and
(5) A statement is a formula only if it is constructed according to rules (1) to (3).

For example, $(\sim((\sim p_2) \wedge q_5))$ is a formula, since it is constructed according to rules (1) to (3).

However, (r ∧ (~p)) is not a formula, since 'r' is not an atomic statement and has not been defined as a formula.

A *production* is of the form: $X_1, X_2, ..., X_n \vdash Y$; where 'X$_1$', 'X$_2$', ..., 'X$_n$', and 'Y' are statements, and if a formula occurs in Y, then it occurs in at least one of $X_1, X_2, ..., X_n$. In such a production, it is said that $X_1, X_2, ..., X_n$ is the *premise* and Y is the *conclusion* of the production.

A *formula value assignment* is of the form P/V, where 'P' is a formula and 'V' is either * or #, or a real number, α. In such an assignment, *V is the value of P*.

A predicate, $\mathcal{P}$, of n arguments is of the form: $\mathcal{P}(p_1, p_2,..., p_n)$; where $p_1, p_2,..., p_n$ are atomic statements which occur in $\mathcal{P}$.

Since at this time we are not concerned with the generation of theorems, but only whether or not a given formula is a theorem, the axioms will not be given at this time. Instead, the following *transformation rules* for truth value analysis are given.

$$P/\#, (\sim P) \vdash (\sim P)/*$$

$$P/*, Q/*, (P \wedge Q) \vdash (P \wedge Q)/*$$

$$P/\alpha, P \vdash P/\alpha$$

where 'P' and 'Q' are either formulas or predicates which are assumed to have the assigned values. It is assumed that any other value assignments of the premise will produce a value assignment of # in the conclusion.

The first seven sets, the seven-tuple $(\mathcal{A},\mathcal{S},\mathcal{O},\mathcal{F},\mathcal{R},\mathcal{V},\mathcal{G})$, define the *signature* of the language. The final four sets, the four-tuple $(\mathcal{T},\mathcal{Q},\,_F\mathcal{V},\mathcal{D})$, define the *extensions* of the language, while all eleven define the *logical language*, $\mathcal{L}$.

In the case of quantifiers, each quantifier is associated with a quantifier type, $\langle n_1,n_2,\ldots,n_n\rangle$; that is, the different objects associated with the theory. 'Type', or 'sort', designates the various classes of a language; for example, components, sets of components, family of sets, etc. are different "types."

If $\mathcal{M}_i$ is a system object-set; for example, $\mathcal{M}_i = \{administrators\}$ or $\mathcal{M}_i = \{teachers\}$ or $\mathcal{M}_i = \{students\}$ or $\mathcal{M}_i = \{textbooks\}$ or $\mathcal{M}_i = \{classrooms\}$ or etc., and $\mathcal{Q}_{\mathcal{M}}$ is the quantification of $\mathcal{M}$, normally designated $\mathcal{Q}w(\mathcal{M}_i(w))$, with respect to $\mathcal{Q}$ in that object-set, then:

$$\mathcal{Q}w\phi(w) \equiv \{w \in \mathcal{M}_i \mid \phi(w)\} \in \mathcal{Q}_{\mathcal{M}i}.\,^{5}$$

The following *Härtig* and *Rescher* quantifiers are examples of the value of this formalization.

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[5] We appreciate the work of Henry who provides this generalization of quantifier at www.PlanetMath.org. Remember that a quantification is but a subset of the power set of the system object-set.

Härtig's quantifier is a quantifier of two variables and two formulas, written $\mathrm{I}wy\phi(w)\psi(y)$. This quantification asserts that the cardinality of two sets that make two functions true is equal:

$$\mathrm{I}wy\phi(w)\psi(y) \equiv (|\{w \mid \phi(w)\}| = |\{y \mid \psi(y)\}|) \in \mathcal{Q}_{\mathcal{M}}$$

That is, the cardinality of the values of w that make $\phi(w)$ true is the same as the cardinality of the values of y that make $\psi(y)$ true.

Rescher's quantifier is written $\mathrm{J}wy\phi(w)\psi(y)$, and asserts $|\{w \mid \phi(w)\}| \le |\{y \mid \psi(y)\}|$, that is:

$$\mathrm{J}wy\phi(w)\psi(y) \equiv (|\{w \mid \phi(w)\}| \le |\{y \mid \psi(y)\}|) \in \mathcal{Q}_{\mathcal{M}}$$

These quantifications can be written with the following more general quantifiers: $\forall$, $\exists$ and ι, where '$\forall$' is the universal quantifier, '$\exists$' is the existential quantifier, and 'ι' designates "that" or "that one," used when there is only one object identified. Using these quantifiers, the above two quantifiers can be written as:

Härtig's quantifier: $\iota\mathcal{M}\forall w\exists\phi\forall y\exists\psi\,\phi(w)\psi(y) \equiv (|\{w \mid \phi(w)\}| = |\{y \mid \psi(y)\}|) \in \iota_{\mathcal{M}}$

Rescher's quantifier: $\iota\mathcal{M}\forall w\exists\phi\forall y\exists\psi\,\phi(w)\psi(y) \equiv (|\{w \mid \phi(w)\}| \le |\{y \mid \psi(y)\}|) \in \iota_{\mathcal{M}}$

These statements can be simplified if we had a class quantifier. Let $\hat{w}$ and $\hat{y}$ be the quantifiers that designate the classes that make a predicate true; for example, the predicate $\hat{w}\hat{y}\mathcal{M}(w,y)$. $\hat{w}$ and $\hat{y}$ designate the classes of all components that make $\mathcal{M}(w,y)$ true. Let $\hat{w},\hat{y}\in \mathcal{Q}$, and $w,y\in\mathcal{M}$, then the above statements can be written as follows:

Härtig's quantifier: $\hat{w}\hat{y}\phi(w)\psi(y) \equiv (|\{w \mid \phi(w)\}| = |\{y \mid \psi(y)\}|) \in \mathcal{Q}_{\mathcal{M}}$

Rescher's quantifier: $\hat{w}\hat{y}\phi(w)\psi(y) \equiv (|\{w \mid \phi(w)\}| \le |\{y \mid \psi(y)\}|) \in \mathcal{Q}_{\mathcal{M}}$

That is, $\hat{w}$ and $\hat{y}$ designate the classes of objects that are elements of $\mathcal{M}$. When, as with the *Härtig's* and *Rescher's quantifiers*, a class may be frequently identified, then a specific quantifier may be appropriate as with I and J which replace the classes $\hat{w}$ and $\hat{y}$.

For *ATIS*, various quantifiers may be developed, in particular, for the *Behavioral Affect Relation Qualifiers*. Whereas the qualifiers define the system activity for a specific property, the quantifiers define the associated relation; that is, the set of all components that make the quantification true with respect to a specific object-set. For *ATIS*, the generalized quantifier is defined as: $\mathrm{A}x_1 x_2 \ldots x_n[\phi(x_1)\phi(x_2)\ldots\phi(x_n)]$.

For example, consider the *Behavioral Affect Relation Qualifier 'Control'* that defines the elements of the affect relation $\mathcal{A}_\mathcal{M}$. This qualifier is defined by the quantifier $A^C(x_1,y_1)(x_2,y_2)\ldots(x_n,y_n)\phi_1(x_1,y_1)\phi_2(x_2,y_2)\ldots\phi_n(x_n,y_n)$; and, for four components, asserts:

$$A^C(x_1,y_1)(x_2,y_2)\phi_1(x_1,y_1)\phi_2(x_2,y_2) \equiv (\{(x_1,y_1) \mid \phi_1(x_1,y_1)\} \cup \{(x_2,y_2) \mid \phi_2(x_2,y_2)\} \in \mathcal{A}_\mathcal{M} \in \mathcal{A})$$

In general, we have the following quantifier, where $\mathcal{X}$ is any system qualifier:

ATIS Quantifier:

$$A^\mathcal{X}(x_1,y_1)\ldots(x_n,y_n)\phi_1(x_1,y_1)\ldots\phi_n(x_n,y_n) \equiv [\{(x_1,y_1) \mid \phi_1(x_1,y_1)\} \cup \ldots \cup \{(x_n,y_n) \mid \phi_n(x_n,y_n)\} \in \mathcal{A}_\mathcal{M} \in \mathcal{A}]$$

This can be written more concisely as: $A^\mathcal{X}\bigcup\{(x_i,y_i) \mid \phi_i(x_i,y_i)\}_{i=1,\ldots,n}\} \in \mathcal{A}_\mathcal{M} \in \mathcal{A}$; or if this quantifier has been defined as representing only a specific relation, then $A^\mathcal{X}\mathcal{M}(x,y)$ would represent the class of all components that make $\mathcal{M}(x,y)$ true.

Now, if the predicate $\mathcal{M}(x,y)$ is a true statement in $\mathcal{M}$, then we say that $\mathcal{M}$ is a model of $\mathcal{M}(x,y)$, and write $\mathcal{M} \vDash \mathcal{M}(x,y)$. Let $\mathcal{U}_\mathcal{M}$ be the universe of all statements that are true in a model, $\mathcal{M}$, then $\forall \mathcal{M}(x,y) \in \mathcal{U}_\mathcal{M}[\mathcal{M} \vDash \mathcal{M}(x,y)]$.

Generalizing this to define the *ATIS Theory Model*, we have a *theory-model production* of the form:

V_1, V_2, ..., $V_n \vDash W$; where 'V_1', 'V_2', ..., 'V_n' are atis-Properties, atis-Axioms, or atis-Statements derived from the atis-Axioms, and 'W' is an atis-Object that occurs in at least one of V_1, V_2, ..., V_n or is derived therefrom by an application of Modus Ponens or Universal Generalization. In such a *theory-model production*, V_1, V_2, ..., V_n is the *model* and W is the *theory* of the *production*.

As a result of this logical language, specific *ATIS Quantifiers* will define specific affect relations. When the affect relations are defined, then the system can be searched for those specific relations, thus identifying the system structure. When a *threshold measure* has been reached, then that affect relation will identify specific system properties. (*Threshold measure* is the measure identified as the minimum affect relation cardinality required for recognition of the affect relation.)

Another advantage of this quantifier for *ATIS* is that now; for example, an education system can be analyzed in terms of all of the *Command* qualifiers for all affect relations, or all of the *Compact Properties*, etc. These quantifiers will provide a measure of these various properties for the system rather than for each affect relation individually. So the *ATIS Quantifiers* can be used to define affect relations, or they can be used to analyze various affect relations for similar characterizations.

With the above discussion, we now have a *formal logic* defined as follows:

Formal Logic =$_\text{df}$ A formal language that contains—

(1) Symbols,

(2) Well-formed formulas derived from the symbols as determined by formation rules,

(3) Axioms that are selected well-formed formulas, and

(4) Transformation rules, normally consisting of two: Modus Ponens and Generalization.

The *Predicate Calculus* can be either first-order or higher-order logic.

In *first-order logic*, quantification covers only individual elements (components) of a specific type or class; that is, only elements of a well-defined set (class) are considered. First-order logic results in verifying properties of a class or subclass of elements.

In *second-order logic*, quantification covers predicates. Second-order logic results in verifying properties of a class or subclass of predicates.

In *higher-order logic*, quantification covers predicate formulas. Higher-order logic results in verifying properties of a class or subclass of predicate formulas.

Whereas the *Sentential* and *Predicate Calculi* provide the logical foundation of the empirical sciences, such application must be done with care when extending that application to *ATIS*. In fact, however, the logic required for *ATIS* is less complex than that required for mathematics and the mathematical sciences, at least initially. The reason is that mathematics and the mathematical sciences must consider distinctions between "x's" that represent "unknown" and "variable" elements. The "unknown" uses are referred to as the "free" occurrences of x, and the "variable" uses are referred to as the "bound" occurrences of x. In *ATIS*, only "bound" occurrences of x will be required.

For this reason, many of the problems encountered by mathematicians relating to the *Predicate Calculus* will not be a problem in the logical analyses of *ATIS*. The reason is that, as noted above, *ATIS* does not consider any statement with free occurrences of x; that is, there are no "unknowns." As will be seen, statements with "unknowns" in *ATIS* are non-sense. For *ATIS* all uses of x are bound; that is, they are variables.

In *ATIS* problems are not being solved in which an unknown is being sought, but what is being sought are the system relations that are true for all described components of a system. The problem with seeking unknowns in the type of statements that are being considered is that it is difficult, if not impossible, to assign any proper meaning to such statements.

For example, the following is a bound occurrence of x:

For $x \in \mathcal{S}$,

$$\forall x (I_p^{\uparrow}(x) \supset {}_\mathcal{S} \mathscr{F}^{\downarrow}(x))$$

However, '$I_p^{\uparrow}(x)$' may or may not make sense when x is an element of just any unknown system, or even within a known system. That is, let '$I_p^{\uparrow}(x)$' be a translation of "x is the increasing input of the toput subsystem." While this English sentence is grammatically correct and has a recognizable meaning, its meaning within *ATIS* is highly suspect, since the x is now an unknown, or simply fanciful. Even if x can be construed as the input of a toput subsystem, x cannot be construed as "increasing" since it is but a single component. Or if it can be construed as increasing, then there are other assumptions of which we are not informed. x in this context is considered an unknown, or is a free occurrence of x. It is a situation in which we would have to determine under what conditions and in which systems this statement would have a proper meaning. Such statements are precluded from *ATIS* analyses.

In order to be selective of our logic, its application must be understood. The types of systems with which we are concerned are *Intentional Systems*. *Intentional Systems* are ones that are goal-oriented, or that have "intended" outcomes. For the analyst of general systems, an *Intentional System* is one that is predictable within certain parameters; that is, its *behavior* is predictable under certain system component relations. The challenge is to determine which system component relations are predictable and what outcomes are obtained as a result of those relations.

The problem of selecting a specific logic on which to base an analysis of general systems is that such systems are *Complex Systems*. *Complex Systems* are systems that are defined by large numbers of components with a large number of heterarchy connections (affect relations) that determine the behavior of the system.

In general, it has been concluded that such systems cannot be analyzed with linear logics, such as logics founded on implication and Modus Ponens, as are the *Sentential* and *Predicate Calculi*. However, such conclusions have been founded on the beliefs that systems cannot be analyzed that have multiple relations. Such is not the case.

Yi Lin[6] has defined systems with multiple relations. It is just such systems that are required for an analysis of *ATIS*. Further, however, the assumption that the *ATIS Predicate Calculus* is linear is misplaced. By reference only, it is recognized that an APT&C analysis has been incorporated into the evaluation of this systems theory, an analysis that is non-linear.

What is required for now is a formal method to analyze general systems, a symbolic logic and mathematical logic that formally express the properties and relations of a system such as system behavior, system structure, dynamic states, morphisms, etc.

The *Sentential Calculus* is frequently defined in terms of truth tables that provide a truth-functional analysis of statements. However, since *ATIS* is defined as an *Axiomatic-General Systems Behavioral Theory*, we will approach both the *Sentential Calculus* and the *Predicate Calculus* as axiomatic theories. Such an approach lends itself to clear statements of theorems and proofs. Further, such axiomatic logics are required since truth-table logics cannot address statements in general, and the complex statements of *ATIS*, in particular.

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[6] Lin, Yi (1999). *General Systems Theory: A Mathematical Approach*. Kluwer Academic/Plenum Publishers, NY.

Symbolic Logic

Symbolic logic is a tool designed for scientific reasoning. In particular, it is a tool designed for *ATIS* reasoning, and also for educology reasoning; such reasoning required for a proper analysis of an *Education Systems Theory* (*EST*).

It is through the use of symbolic logic that theories are made precise and explicated. The main reason for using symbolic logic is that it is a means of obtaining precise definitions for the logical consequence of one statement from another. The main advantage of a formal logic is in being able to prove statements about a theory.

One of the greatest advantages of a formal logic is that it provides a precise definition for determining when one statement is a logical consequence of another.

The logical consequence of one statement from another is obtained by a sequence of well-defined statements such that each statement is known to be valid; that is, is an axiom, is an assumption or is derived from previous statements of the sequence according to specific rules of inference.

Valid statements are only those that are axioms or are derived only from axioms. Rules of inference are restricted to Modus Ponens and Universal Generalization.

The *ATIS Sentential Calculus* is a theory of statement formulas in which the statements are translations of sentences within *ATIS*. For *ATIS*, a statement is a declarative sentence that relates exclusively to system components, relations or properties of *ATIS*. While the *Sentential Calculus* herein considered may be equivalent to that used for mathematics and the mathematical sciences, it is important to note that the extended logic herein considered is that developed specifically for *ATIS*, and is not intended to be a "mathematical logic" for mathematicians, but it is a mathematical logic designed specifically for *ATIS*.

Statements will be expressed by capital letters; e.g., "P," "Q," etc., and are translations of their English sentences "A" and "B," respectively. All statement functions of the theory are derived from only two undefined functions: '∧' and '∼', which are read "and" and "not," respectively.

Therefore, 'P ∧ Q' is read "P and Q," and is a translation of the English sentence "A and B"; and '∼P' is read "not P," and is a translation of the negation of the English sentence "A." While we will read '∼P' as "not P," the English sentence may take several forms depending on what is required to assert the negation of "A."

A statement formula is a string of statements combined with ∧ and ∼. '∧' and '∼' are the first two functions of the *Sentential Calculus*:

(1)   P ∧ Q

(2)   ∼P

While these functions are undefined, they will be interpreted as having "truth values" defined by the following "truth-value tables." While the values are commonly thought of as "True" or "False," in fact they are but assigned values with no relation to "truth." This will be emphasized in the following tables by using '⊤' for "true" and '⊥' for "false." The truth table simply presents the possible combinations of '⊤' and '⊥'.

Table 1: Truth table for the operation '∧'

P	**Q**	P	∧	Q
⊤	⊤	⊤	⊤	⊤
⊤	⊥	⊤	⊥	⊥
⊥	⊤	⊥	⊥	⊤
⊥	⊥	⊥	⊥	⊥

Table 2: Truth table for the operation '∼'

P	**∼P**
⊤	⊥
⊥	⊤

Having demonstrated that the values of these operations are dependent only on the form of the statement, we will now revert to the commonly used notation of "T" and "F" for the values of the "truth" tables. As demonstrated in Tables 1 and 2, the operation '∧' takes the value '⊤' (or "T") only when both P and Q are ⊤; and the operation '∼' takes the value that is the alternative to P.

By convention, 'P ∧ Q' may be written as 'PQ'.

'P ∨ Q' ("P or Q"—inclusive "or"; i.e., and/or), 'P ⊻ Q' ("P or Q"—exclusive "or"; i.e., not both), 'P ⊃ Q' ("P implies Q" or "If P then Q"), and 'P ≡ Q' ("P if and only if Q" or "P is equivalent to Q") are defined as follows:

(3) $P \vee Q =_{df} {\sim}({\sim}P{\sim}Q)$

(4) $P \veebar Q =_{df} {\sim}({\sim}P{\sim}Q) \wedge {\sim}(PQ)$

(5) $P \supset Q =_{df} {\sim}(P{\sim}Q)$

(6) $P \equiv Q =_{df} {\sim}(P{\sim}Q) \wedge {\sim}({\sim}PQ) =_{df} (P \supset Q)(Q \supset P)$

These six functions are the ones by which the *Sentential Calculus* is explicated.

Since implication, ⊃, will be a very important function of the *ATIS Sentential Calculus*, its interpretation will be further considered. The function 'P ⊃ Q' may be read in any one of the following ways, all of which are equivalent:

> Q is a necessary condition for P,
>
> P is a sufficient condition for Q,
>
> Q if P,
>
> P only if Q,
>
> P implies Q, and
>
> If P then Q.

Consider a list of statements, $P_1, P_2, \ldots, P_n$. Combine these statements by the use of '∧' and '~' in any manner desired, and call the result 'Γ'. As a result of this construction of Γ, Γ will be called a *statement formula*. A statement formula that is written using only '∧' and '~' will be defined as being in "standard form." The purpose of the *Sentential Calculus* is to determine when a statement formula is valid, and validity will be determined when the statement formula is "true."

One means for determining when statement formulae are true, is by the use of "truth tables." Truth tables can assist in determining when a statement formula is true regardless of the meaning of the statements that make up the formula. That is, in general, if the truth of the statements of a formula is unknown, then the truth of the formula cannot be determined.

However, statement formula validity can be determined, regardless of the validity of the statements, when the statement formula has a certain structure. For example, the statement formula "~PP" is always false and ~(~PP) is always true regardless of the validity of P. Truth tables can assist in determining under what conditions a statement formula is valid in the Sentential Calculus. (See most any basic text on logic for a discussion of truth tables.)

Axiomatic Sentential Calculus

Whereas truth tables are convenient for determining the validity of statement formulas, such tables cannot be generalized to all statements. To date, only an axiomatic method is known that is able to obtain validations of general statements. As a transition to the axioms required for validation of general statements, we will first consider a subset of those axioms, the truth-value axioms. These axioms will provide an excellent transition to axiomatic logic, since these axioms will produce those statements considered earlier, the statement formulae that can be validated by use of a truth table, and, therefore can be easily validated by two methods—truth tables and axioms.

In general, the axiomatic definition of valid statements is obtained by the following process: (1) Certain selected statements are called 'axioms' (and their selection may be somewhat arbitrary and may be modified to achieve certain objectives); (2) A transformation rule is selected, normally Modus Ponens (although other transformation rules are possible; Generalization or Modus Talens); and (3) 'Valid statements' are those statements that are either axioms or can be derived from two or more axioms by successive applications of Modus Ponens.

It is worth mentioning again that only the form of the statements and not their meaning determines valid statements.

There are three axioms of the *Truth-Value Sentential Calculus* and one logical rule.[7]

Let 'P', 'Q', and 'R' be statements of the theory, then—

The logical rule is Modus Ponens: P, P $\supset$ Q $\vdash$ Q; and the axiom schemas are:

(1)   P $\supset$ PP

(2)   PQ $\supset$ P

(3)   P $\supset$ Q .$\supset$. ~(QR) $\supset$ ~(RP)

There are an infinite number of statements that will comprise the truth-value axioms, however, all axioms will be of one of the above three general forms, the axiom schemas. Further, all theorems of the *Truth-Value Sentential Calculus* can be derived from these three axioms and Modus Ponens.

A theorem will take the form: P_1, P_2, …, P_n $\vdash$ Q, where the P's are statements and Q is an axiom, or Q is one of the P's, or Q is derived from the P's by repeated applications of Modus Ponens.

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[7] Much of the logical foundations in this report are a result of studies derived from *Logic for Mathematicians* by J. Barkley Rosser, McGraw-Hill Book Company, Inc., New York (1953). The logical development contained herein is an extension of that work.

'⊢' is read "yield", or in the case when we have only '⊢ Q' it is read "yields Q."

The theorem $P_1, P_2, \ldots, P_n \vdash Q$ indicates that there is a sequence of statements, $S_1, S_2, \ldots, S_m$, called the *proof* of the theorem, such that S_m is Q and for each S_i, either:

    (1) S_i is an axiom,

    (2) S_i is a P; i.e., an assumption,

    (3) S_i is the same as some earlier S_j, or

    (4) S_i is derived from two earlier S's by Modus Ponens.

The sequence $S_1, S_2, \ldots, S_m$ is a proof within the Symbolic Logic so that Q is logically derived from the assumptions $P_1, P_2, \ldots, P_n$.

The following theorems are readily provable concerning ⊢:

Theorem. If $P_1, \ldots, P_n \vdash Q$, then $P_1, \ldots, P_n, R_1, \ldots, R_m \vdash Q$.

    Proof: Let $S_1, \ldots, S_s$ be the proof of $P_1, \ldots, P_n \vdash Q$ where S_s is Q. Clearly that same sequence will yield Q regardless of any additional assumptions.

Theorem. If $P_1, \ldots, P_n \vdash Q_1$ and $Q_1, \ldots, Q_m \vdash R$, then $P_1, \ldots, P_n, Q_2, \ldots, Q_m \vdash R$.

Theorem. If $P_1, \ldots, P_n \vdash Q_1, R_1, \ldots, R_m \vdash Q_2$, and $Q_1, \ldots, Q_q \vdash S$, then

$$P_1, \ldots, P_n, R_1, \ldots, R_m, Q_3, \ldots, Q_q \vdash S.$$

Theorem. If $\vdash Q_1$ and $Q_1, \ldots, Q_m \vdash R$, then $Q_2, \ldots, Q_m \vdash R$.

Theorem. If $\vdash Q_1, \vdash Q_2, \ldots, \vdash Q_m$ and $Q_1, \ldots, Q_m \vdash R$, then $\vdash R$.

Since our main concern is to provide the means to explicate *ATIS*, the *Sentential Calculus* will not be further explicated. The following *List of Logical Schemas* is provided to facilitate the explication of the theories. This list is not exhaustive, but does represent those schemas that lend themselves to a fruitful explication of the theories. Following this list, the schemas will be used to demonstrate the value of such a symbolic logic by providing proofs of theorems. It is noted that technically these schemas are not actually part of the *Sentential Calculus* but are part of the metatheory, the *Meta-Sentential Calculus*. They are statements about the calculus that define the form or schemas that the theorems of the theory actually take.

List of Logical Schemas

The following list of the logical schemas is provided to facilitate the proof of theorems.

The "*System Construction Theorems*" (*SCTs*), derived directly from the axioms of the *Sentential Calculus* and the intuitive creativity of a researcher or interpreter, provide a means of developing the connectedness of a system or of determining predictive outcomes. These should prove important in developing the system topology. The significance and use of *SCTs* will be clarified before presenting the logical schemas.

Significance of SCTs

System Construction Theorems provide the means to develop, enhance or further the explication of a theory. The significance is that they provide additional statements than what are found in the assumptions. Since they are statements of the theory, however, they are valid statements, they are not just any statements whimsically selected.

They may, however, be statements that are intuitively derived and thereby declared to be valid statements of the theory. As an initial example, however, consider the case where the derived statement is an axiom. As an axiom, it is a valid statement of the theory. Consider Logical Schema 3: $P \supset R \vdash P \supset (Q \supset R)$. Let Q be Axiom 105 of SIGGS, "If centrality increases, then toput decreases." Then, regardless of what P and R represent, the following is valid:

$P \supset R \vdash P \supset$ ("If centrality increases, then toput decreases" $\supset$ R), where P and R are statements of the theory and $P \supset R$ is assumed to be true. For example:

Let P = "System complete connectivity increases"; and R = "System feedin increases." Then, $P \supset R$ is a statement of Axiom 100; and, therefore $P \supset R$ is true. Then, from our theorem we have:

> "System complete connectivity increases" $\supset$ "System feedin increases" $\vdash$
>
> > "System complete connectivity increases" $\supset$ ("If centrality increases, then toput decreases" $\supset$ "System feedin increases").

The conclusion of this statement is equivalent to the following:

> > "System complete connectivity increases" $\supset$ ("centrality decreases or toput decreases" $\supset$ "System feedin increases").

It is probably clear that this is a non-obvious theorem; hence the value of the formal logic is established. But, what does it tell us?

This theorem provides a means to control a system. If the target system has complete connectivity increasing and system feedin increasing then the assumption of the theorem is satisfied. Now, assume that the target system is a terrorist system and that it is desired to decrease the complete connectivity. One way to accomplish this is to decrease toput and feedin. By decreasing toput and feedin under these conditions, system complete connectivity will decrease. Further, decreasing toput decreases feedin. Therefore, only one factor, toput, has to be controlled in order to achieve the objective of decreasing complete connectivity.

This analysis demonstrates several points. First, there are numerous non-obvious theorems that can be derived from a logical axiomatic analysis of the theory. Second, some of the outcomes, as with the above theorem, are counter-intuitive. In this case, the measure of complete connectivity is dependent on the potential complexity of the system, such complexity being degraded when toput is reduced. Third, the *SCTs* provide a fruitful means to analyze a system, but may require the intuitive skill of the analyst. On the other hand, where the logic is required for applications similar to SimEd, by defining certain "replacement" or "substitution" rules that will allow for selection of various properties or newly acquired data such logic can be programmed. These rules will probably have to be developed by an analyst who has a grasp of the pragmatic content of the theory.

Logical Schemas

Logical Schema 0:	$P \supset Q, Q \supset R \vdash P \supset R$	(Transitive Property of $\supset$)
Logical Schema 1:	$P \supset Q, R \supset Q \vdash P \vee R \supset Q$	
Logical Schema 2:	$P \supset Q, R \supset S \vdash PR \supset QS$	
Logical Schema 3:	$P \supset R \vdash P \supset (Q \supset R)$	(System Construction Theorem Schema)
Logical Schema 4:	$P \supset Q, P \supset R \vdash P \supset QR$	
Logical Schema 5:	$\vdash Q \supset P .\equiv. \sim P \supset \sim Q$	
Logical Schema 6:	**If** $P \supset Q$, **then** $P \vdash Q$; **and If** $P \vdash Q$, **then** $P \supset Q :\equiv:$	

$$P \vdash Q .\equiv. \vdash P \supset Q$$

"$P \vdash Q .\supset. P \supset Q$" is the ***Deduction Theorem***.

Logical Schema 7:	$\vdash \sim(\sim PP)$	
Logical Schema 8:	$\vdash \sim\sim P .\equiv. P$	
Logical Schema 9:	$\vdash \sim P \vee P$	
Logical Schema 10:	$P \vdash Q \supset PQ$	(System Construction Theorem Schema)
Logical Schema 11:	$\sim(QR) \vdash R \supset \sim Q$	
Logical Schema 12:	$P \supset Q \vdash PR \supset QR$	(System Construction Theorem Schema)
Logical Schema 13:	$R \supset S \vdash PR \supset PS$	(System Construction Theorem Schema)
Logical Schema 14:	$PQ \supset P \vdash P \supset (Q \supset R)$	(System Construction Theorem Schema)
Logical Schema 15:	$\vdash PQ \supset R .\equiv. P \supset (Q \supset R)$	
Logical Schema 16:	$P \supset \sim Q \vdash P \supset (Q \supset R)$	(System Construction Theorem Schema)
Logical Schema 17:	$P \supset \sim R \vdash P \supset \sim(QR)$	(System Construction Theorem Schema)
Logical Schema 18:	$P \supset Q, P \supset \sim R \vdash P \supset \sim(Q \supset R)$	
Logical Schema 19:	$P, P \supset Q \vdash Q$	(Modus Ponens)
Logical Schema 20:	$\sim Q, P \supset Q \vdash \sim P$	(Modus Talens)

The $\mathcal{ATIS}$ Predicate Calculus

While the *Sentential Calculus* has been presented so as to demonstrate the usefulness of a formal logic, the $\mathcal{ATIS}$ *Predicate Calculus* will be only briefly presented with what is required to understand its application to the analysis of the target theories. Unlike the *Sentential Calculus*, however, it is important to note that this predicate calculus is distinctly different from that required for mathematics or the mathematical sciences. Without going into any great discussion, the reason is that for $\mathcal{ATIS}$ only bound occurrences of x are considered since free occurrences do not have any apparent meaning within $\mathcal{ATIS}$.

It was previously stated that the difference between the *Sentential* and *Predicate Calculi* was that the *Sentential Calculus* is concerned with the form of the aggregate statement with no concern of what is contained within the statement, whereas the *Predicate Calculus* is concerned with the logic of predicates; that is, statements and their constituent parts, as related to quantifiers—normally the universal and existential quantifiers. This extension will now be considered.

To make the transition from the *Predicate Calculus* required for the mathematical sciences and that required for $\mathcal{ATIS}$, we will first consider the predicate notation. The predicate notation will take the form of a function; e.g., P(x), where 'x' is an "unknown." If we can prove that P(x) is true for the unknown 'x', then we have ⊢ P(x). If we have ⊢ P(x) then we can replace 'x' with a variable and will conclude: ⊢ ∀xP(x). For *ATIS*, it is assumed that all predicates are bound, and, therefore, all occurrences of x are variables and the truth-value of all predicate functions can be determined. Therefore, with respect to any occurrence of x, the task is to assert ⊢ ∀xP(x) and determine if a proof exists.

Since Alonzo Church, in 1936, proved that there is no decision procedure for the *Predicate Calculus*, then the only affirmative conclusion that is possible concerning ⊢ ∀xP(x), with respect to the *Predicate Calculus*, is that it is true. If no such affirmative conclusion can be found, then nothing more can be said concerning the validity of the statement within the theory. Further, the conclusion is even stronger. Church proved that there is no decision procedure regardless of what axioms are considered.[8]

This is great news for the logician and for any researcher or analyst who is attempting to evaluate $\mathcal{ATIS}$ or an *EST*. What Church has proved is that there will always be a need for the researcher and analyst, since the Predicate Calculus, and the $\mathcal{ATIS}$ *Predicate Calculus*, in particular, has no decision procedure, and, therefore, cannot be fully programmed. It is not asserted that the $\mathcal{ATIS}$ *Predicate Calculus* cannot be partially programmed, because it can, but it cannot be completely programmed. The part that can be programmed, as seen below, is that part that results from the axioms that define the $\mathcal{ATIS}$ *Sentential Calculus*.

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[8] This is not to say that when empirically tested as a hypothesis the assertion may be shown to be "false." That is a different matter. However, no such conclusion can be derived from the axioms.

This point is worth elaborating. The scope of the programmable theory must be determined, since such programs will clearly make the theory and any of its proprietary software products more appealing to users. As reflected by the extensive list of theorems that can be derived from the *ATIS Sentential Calculus*, numbering in the tens-of-thousands, and the numerous *Theorem Schemas* cited previously, it is seen that a very fruitful analysis of a system can be obtained.

Further, it is proposed that utilizing data mining technologies will extend the value of this fruitful analysis even further. That is, the theory software can be used as an interpreter of the data mining structured outcomes, thus enhancing the time-sensitive results required in a terrorist environment, and possibly in an educational environment. With this technology, it is no longer required that one must wait for a pattern to be determined by the data mining, but that the theory analysis will enhance the ability of the data mining technology to recognize patterns and predicted terrorist behavior or targets much earlier than utilizing the data mining technology alone.

That said, it must also be recognized that when evaluating a specific system, the logic is only semi-decidable. That is, an analyst can affirmatively determine, *within the theory*, that a theorem is true, but cannot, under any circumstances, prove that it is false. The reason for this is three-fold: (1) As soon as an empirical system is recognized, the problem for the analyst reverts to considerations within the *ATIS Predicate Calculus*; (2) Church has proved that such considerations are only semi-decidable; and (3) The reason that such problems considered in an empirical system are only semi-decidable is that one never knows if all possibilities have actually been considered in the proof. Systems, especially behavioral systems, are complex. This must be recognized and recognized as something positive. That is, the researcher and analyst have some very difficult tasks confronting them.

So, is the analyst without recourse? Not at all. Creative proofs from outside the theory are possible. If a reasoned argument can be found that can be construed as part of the logic of the meta-theory, then a particular theorem can be cited as being invalid. Once the theorem has been proved in the meta-theory as being invalid, one is then justified by claiming that the theorem is invalid within the theory. The significance of this is that the results of this proof can then be inserted into the theory as though it had been proved within the theory.

One is cautioned not to insert theorems directly into a computer program that has been developed as a model of the theory. However, that precaution is with respect to the *ATIS Sentential Calculus*. The *ATIS Predicate Calculus* is an entirely different matter. Whereas the theorems of the *ATIS Sentential Calculus* can be obtained directly from the axioms and, therefore, do not warrant the arbitrary insertion of theorems, the same cannot be said for the *ATIS Predicate Calculus*.

Further, there will be additional work for the researcher and analyst once the initial logic has been developed and implemented for theory model applications. There are additional analyses that can be made with respect to empirical systems. It is intended that the structural properties of a system can be recognized as the topology of the system, and that the power of mathematical topology can be modified, as the predicate calculus has been, in such a way that the power of a modified mathematical topology can be used to assist in the analysis of a system. Such analyses may also have to be performed by a researcher or analyst directly with little or no reliance on a computer program. These results also will have to be manually inserted into any computer program that has been designed for a particular system.

What this simply means is that researchers and analysts of behavioral systems will always have a job. To this researcher, that is something to look forward to.

As noted previously, due to the nature of the target theories, there will be no need to distinguish between "free" and "bound" occurrences of 'x', since, without any loss of generality, all occurrences of 'x' are considered to be bound. In view of this, we have the following axioms:

(1) $P \supset PP$

(2) $PQ \supset P$

(3) $P \supset Q :\lor: \sim(QR) \supset \sim(RP)$

(4) $\forall x(P \supset Q) :\supset: \forall xP \supset \forall xQ$

(5) $P \supset \forall xP$

(6) $\forall xP(x,y) \supset P(y,y)$

It should be recognized that the first three axioms are simply taken from the *Sentential Calculus*; that is, all such resulting theorems are still valid in the *Predicate Calculus*.

As seen from the axioms, the only quantifier is the universal quantifier. The existential quantifier will be defined in terms of the universal:

$$\exists xP =_{df} \sim\forall x \sim P$$

From this definition, we have the following equivalences:

$\vdash \exists xP \equiv \sim\forall x \sim P$

$\vdash \forall xP \equiv \sim\exists x \sim P$

$\vdash \sim\exists xP \equiv \forall x \sim P$

$\vdash \sim\forall xP \equiv \exists x \sim P$

There are special conditions for ∃ for which additional notations are desired. These are the conditions in which there is exactly one x for which P is true and when there are n x's for which P is true. These notations are as follows:

$\exists^1 x F(x)$ denotes that there is only one x for which $F(x)$ is true; and

$\exists^n F(x)$ denotes that there are exactly n x's for which $F(x)$ is true.

In addition to the two quantifiers, ∀ and ∃, there are two additional quantifiers, one that will be used to specify a single component and one that will specify a class of components. These quantifiers are the descriptor quantifier, ι, and the class quantifier, ŵ. '$\iota x P(x)$' is read, "the x such that P(x)"; and '$\hat{w}F(w)$' is read "the class of w determined by $F(w)$." These are defined as follows:

$\iota x F(x) =_{df} \exists^1 x F(x)$; and

$\hat{w}F(w) =_{df} \iota\alpha\forall w(w\in\alpha \equiv F(w))$

'$\iota x F(x)$' is the *name* of the unique object that makes $F(x)$ true.

An important clarification needs to be made concerning the meaning of 'quantifier'. A logical quantifier designates a qualification of a class by indicating the logical quantity; that is, the specific components to which the qualification applies. 'P' or '$F(x)$' is the *scope* of the quantification; that is, the scope of what is qualified. This will be a frequently used concept in the analysis of systems. The 'Logistic Qualifiers' are those predicates that will be used to quantify a specific set. For example, *Toput* becomes *Input* as the result of quantifying *Toput* with respect to the *Logistic Qualifiers*. This system transition function is defined as follows:

$$\sigma{:}(T_p \times \mathscr{L}_{i=1:n}(F_i(w\in T_p)) \to I_p) = \hat{w}_{I_p}F(w_{T_p})$$

where 'i=1:n' designates "i varies from 1 to n," and '$F_i(w\in T_p)$' is a qualifying statement in $\mathscr{L}$ with respect to w in T_p.

'$\hat{w}_{I_p}F(w_{T_p})$' designates the *Input Class* determined by the *Toput Class* qualified by the $F(w)$'s in $\mathscr{L}$ that make $F(w_{T_p})$ true.

An equivalent notation for ŵP is {w | P}, which is frequently used in mathematics.

Theory Building

In the preceding sections, the need and requirements for an axiomatic logic have been presented. In that discussion the problems relating to theory building that relies on induction, hypothetico-deductive and grounded methodologies were discussed. Now we will consider some specific concerns relating to theory building itself. What follows will be a discussion of several specific points and how to determine if theory building is actually being pursued, and if it is, what one must look for in that theory building and how to validate the theory once it is developed.

First, we will consider how to determine if the validation of a hypothesis is theoretically sound. The basic test is simply to ask the following question:

> Was the hypothesis derived from a theory that is consistent, comprehensive and complete; and, if so, is the theory axiomatic?

With respect to the requirement that the theory be axiomatic, it is simply a recognition that only axiomatic theories have been found to provide the rigorous analyses required to obtain confidence in the theory results. If an axiomatic theory cannot be obtained, then the results can always be questioned either with respect to the validation process or with respect to the "underlying assumptions" that are not stated in the theory. Descriptive and statistical-based theories can never be individually predictive and any results can always be questioned with respect to the descriptive theory, and statistical-based theories, by definition, can never be individually predictive.

Put another way, simply ask yourself:

> Was the hypothesis derived from theory? If so, what is it?

Once the theory has been established, then the next question that needs to be addressed concerns the logical basis of the theory. Most often it will be founded on a predicate calculus. If so, then there are additional questions that relate to that logic.

Any theorems that are derived from the predicate calculus are a result of the form of the theorems and not their content. The theorems of the theory that are derived directly from the basic logic are true because of their logical structure, and not at all because of their content.

In addition to theorems that are derived from the basic logic, there will be theorems that are derived from *ATIS* axioms. Further, there will be theorems that are derived from the axioms obtained as a result of the specific empirical system being considered. Axioms and theorems from the latter two will depend upon the meaning of the terms employed within the theory or system, and not due only to their logical structure.

Class Calculus

Before considering the axioms of *ATIS* and how to develop axioms for specific systems, the axioms of the *Predicate Calculus* will be extended to include the *Class Calculus*. For this extension, a more precise and formal development of the basic logic will be presented so that a clear definition of *term*, *statement* and *formula* can be obtained.

Stratified statements determine classes. However, for *ATIS*, the initial partitioning of the system components and the definition of the system affect relations determine the stratification. Affect relations are, by definition, one class or type higher than the system components, and there is, therefore, no confusion of types.

A statement is determined by the following symbols:

$$\sim \ \wedge \ \forall \ \iota \ \wedge \ \in \quad \textit{``variables:''} \ x_1 \ x_2 \ \dots \ x_n \ x \ y \ \alpha \ \beta \qquad \textit{``statements:''} \ \mathcal{P} \ \mathcal{Q}$$
$$\mathcal{R} \ \mathcal{F}(x) \ \mathcal{F}(y) \ \mathcal{F}(\iota yQ) \ \mathcal{P}(x,y) \ \mathcal{P}(y,y)$$

Following are the definitions of 'term', 'statement' and 'formula'. Due to their use in *ATIS*, all variables are *bound*.

(1) *term* $=_{df}$

    (i) $x_1 \ x_2 \ \dots \ x_n \ x \ y$

    (ii) $\iota x \mathcal{P}$

    (iii) $\hat{w} \mathcal{P}$

(2) *statement* $=_{df}$

    (i) $A \in B$, where 'A' and 'B' are terms, and 'A' is a *component* and 'B' is a *class*, since only sentences concerned exclusively with classes are considered to be statements

    (ii) $\forall x \mathcal{P}$, where 'x' is a variable and '$\mathcal{P}$' is a statement

    (iii) $\sim \mathcal{P}$, where '$\mathcal{P}$' is a statement

    (iv) $\mathcal{P} \wedge \mathcal{Q}$, where '$\mathcal{P}$' and '$\mathcal{Q}$' are statements

(3) *formula* $=_{df}$

    (i)   S, where 'S' is comprised of a sequence of statements constructed with '$\sim$' and '$\wedge$'

(4) $\alpha = \beta =_{df} \forall x(x \in \alpha \equiv x \in \beta) =_{df} \alpha = \beta =_{df} \alpha =_x \beta$

'$\mathcal{P}$' is referred to as the scope of the quantifiers.

Definition (4) defines equality of sets. The last notation, $\alpha =_x \beta$, is very useful in *ATIS*. Due to the complexity of systems, it may be that various properties are defined with respect to the same set of components. Rather than having to consider numerous sets, the properties can be defined with respect to a specific subset. For example, one may wish to determine the behavior of a system with respect to various subsets. Such can be designated as follows: $\mathcal{S} =_B \mathcal{V}$; $\mathcal{S} =_C \mathcal{L}$; and $\mathcal{S} =_D \mathcal{G}$. Then an APT&C analysis can be performed on the following set: $\ell = \{\mathcal{V}, \mathcal{L}, \mathcal{G}\}$.

With the foregoing definitions, we now have the following axiom schemas extended from the *Predicate Calculus* to include the axiom schema for the *Class Calculus*, Axiom (12).

Transformation Rule, Modus Ponens: $\mathcal{P}, \mathcal{P} \supset Q \vdash Q$

(1) $\mathcal{P} \supset \mathcal{P}\mathcal{P}$

(2) $\mathcal{P}Q \supset \mathcal{P}$

(3) $(\mathcal{P} \supset Q) \vee [\sim(Q\,R) \supset \sim(R\mathcal{P})]$

(4) $\forall x(\mathcal{P} \supset Q) \supset (\forall x\mathcal{P} \supset \forall x\,Q)$

(5) $\mathcal{P} \supset \forall x\mathcal{P}$

(6) $\forall x\,\mathcal{P}(x,y) \supset \mathcal{P}(y,y)$

       i. $\forall x F(x) \supset F(y)$

(7) $\forall x,y,z[(x = y) \supset (x \in z \supset y \in z)]$

(8) $\forall x_1, x_2, \ldots, x_n(\forall x F(x) \supset F(\iota y\,Q))$

(9) $\forall x_1, x_2, \ldots, x_n[\forall x(\mathcal{P} \equiv Q) \supset (\iota x \mathcal{P} = \iota x\,Q)]$

(10) $\forall x_1, x_2, \ldots, x_n[\iota x F(x) = \iota y F(y)]$

       i. $\forall x_1, x_2, \ldots, x_n[\iota x \mathcal{P} = \iota y\,Q]$

(11) $\forall x_1, x_2, \ldots, x_n[\exists_1 x \mathcal{P} \supset (\forall x[\iota x \mathcal{P} = x \equiv \mathcal{P}])]$

       i. $\forall x_1, x_2, \ldots, x_n[\exists_1 x F(x) \supset (\forall x[\iota x F(x) = x \equiv F(x)])]$

       ii. $\forall x_1, x_2, \ldots, x_n[\exists_1 x F(x) \supset (\forall y[\iota x F(x) = y \equiv F(y)])]$

(12) $\exists y \forall x(x \in y \equiv \mathcal{P})$

Axiom Schemas (1) to (3) are the truth-value axioms of the Sentential Calculus.

Axiom Schemas (4) to (6) allow for generalization from ⊢$\mathcal{P}$ to ⊢∀x$\mathcal{P}$.

Axiom Schema (6) provides for substitution of a value for a variable.

Axiom Schema (7) allows for substitution of equivalent terms resulting from equality.

Axiom Schemas (8) to (11) are the axiom schemas for ι.

Axiom Schema (8) asserts that if F(x) is true for all x, then ιy$\mathcal{Q}$ is the name of one of those objects.

Axiom Schema (9) asserts that if P and Q are equivalent for all x, then ιx$\mathcal{P}$ and ιx$\mathcal{Q}$ are names of the same object.

Axiom Schema (10) allows for change of variables.

Axiom Schema (11) asserts that if there is a unique x that makes $\mathcal{P}$ true, then ιx$\mathcal{P}$ is that x.

Axiom Schema (12) is the schema that introduces classes. This axiom allows for the *Set Calculus* to be integrated into the formal theory.

Relation Calculus

For *ATIS*, we are concerned with attempting to use as many mathematical constructs as possible while clearly describing the desired system properties.

While mathematics is frequently concerned with *functions*, for *ATIS* the concerns may be directed more toward *relations*.

However, while functions are normally considered as being single-valued, many-valued functions are possible. The relation $[\![x,x^{1/2}]\!]$ is a multi-valued function. $[\![x,x<y]\!]$ also is a multi-valued function. These are well-defined functions since the ordered pairs that define the functions are well-defined. Whether or not these are considered functions or relations is not clear; that is, there does not seem to be any clear distinction between the two. With 'function' being restricted to *single-valued functions*, these examples would be considered as *relations*. One distinction has been that 'function' was restricted to relations that resulted in well-defined curves, whereas 'relation' would be for those statements that defined all other characterizations. Thus, $[\![x,x^{1/2}]\!]$ would be a *function*, and $[\![x,x<y]\!]$ would be a *relation*.

In *ATIS*, the distinction between 'function' and 'relation' will not be considered. The only question is whether or not the appropriate mathematical construct clearly portrays the system characteristic being considered. It appears as though most of the concerns for *ATIS* will be with respect to *morphisms*; that is, *relational mappings*. Whether such mappings are 'functions' or 'relations' is moot. If a single-valued function is required, then such can be stated. For purposes of analysis, morphisms or relations will be considered, since functions are a special type of relation. Further, where the "function notation" is used, it is not to be construed as restrictive. Normally, it will probably designate a single-valued function, but such in this theory is not required. Either the context or by definition, the type of function will be determined.

The *Relation Calculus* for *ATIS* is concerned with the affect relations that define a system and the morphisms that characterize the properties of the system as derived from those affect relations.

The *Relation Calculus* axiom schemas will be presented first. This will complete the presentation of the formal logic.

Following the presentation of the formal logic, the content required for a General System Theory will be introduced. First, the axiom that asserts the existence of a General System will be introduced. Then the axioms that establish the empirical systems that are to be analyzed and the criteria for such analysis will be given.

We have already introduced the notation that will be used to identify a class or set of objects, or components, $\hat{w}F(w)$. Now the characterization of those components will be discussed.

Whereas x identifies a single component within the set, it may be that we wish to identify an object that consists of two or more components. The following notations will be used to identify such sets.

'$\{x, y\}$' identifies a component of a set that consists of two single components.

If it is desired to specify that the set consists only of binary-components, then the following notation will so indicate: $\hat{w}^2F(w)$. This notation designates that the class or set of components consists only of sets each of which contains two single components.

Hence, '$\hat{w}^2F(w)$' designates a *family* of binary sets.

By extension, '$\hat{w}^nF(w)$' designates a family of sets, each member of which contains n components; that is, $\{x_1, x_2, \ldots, x_n\}_i \in w$.

For affect relations, an additional type of set will be required. This set will contain binary-components and a set that contains one of the binary-components in a unary-component set. That is, the set will be configured by the following representation:

$$\{\{x\}, \{x, y\}\}$$

where each unary-, binary-component set is included and no other sets are included. Where there is no confusion, this notation is frequently represented by the ordered pair: (x,y).

For this set, the class quantifier will be represented as: $\hat{w}^{1|2}F(w)$.

By extension, '$\hat{w}^{1|\ldots n-1|n}F(w)$' designates a family of sets that include all and only those ordered subsets of the largest set. For n = 4, the family of sets would be characterized by components of the form: {{a}, {a,b}, {a,b,c}, {a,b,c,d}} = (a,b,c,d), an *ordered* 4-tuple.

With the foregoing definitions, we now have the following axiom schema for the *Relation Calculus*, Axiom (13), which introduces relations.

Axiom (13) $\exists z \forall x,y(\{x,y\} \in z \wedge \{x\} \in z \equiv \mathcal{R})$

The following axiom schemas provide for substitution within and identification of relations.

Axiom (14) $\forall x \mathcal{R}(x) \supset \mathcal{R}(\iota y \mathcal{Q})$

Axiom (15) $\forall x(\mathcal{R} \equiv \mathcal{Q}) \supset (\iota x \mathcal{R} \equiv \iota x \mathcal{Q})$

Axiom (16) $\iota(x,y)\mathcal{R}(x,y) = \iota(p,q)\mathcal{R}(p,q))$

Axiom Schema (14) asserts that if $\mathcal{R}(x)$ is true for all x, then $\iota y \mathcal{Q}$ is the name of one of those relations.

Axiom Schema (15) asserts that if relations $\mathcal{R}$ and $\mathcal{Q}$ are equivalent for all x, then $\iota x \mathcal{R}$ and $\iota x \mathcal{Q}$ are names of the same relation. This is a critical axiom for determining morphisms.

Axiom Schema (16) allows for change of variables.

ATIS Calculus

In the preceding sections, the formal logic has been established.

Now, the calculus must be developed that begins to provide the substance for the desired *ATIS* theories. These theories are descriptive of what will be called *General Systems*. However, the first axiom will introduce the characteristics of 'System'.

The development of the calculus that results in the empirical theory is dependent upon the concept of an *Options Set*. The *Options Set* is that listing of *Properties* and *Associated Axioms* that will result in a system-descriptive theory that will be analyzed pursuant to the derived formal logic

herein established. The specific *Options Set* herein developed is the *ATIS Options Set*. This set will consist of the derived list of properties, and all axioms that are associated with those properties.

An analysis of a system is obtained by determining those properties that are descriptive of the system. Those properties are then identified in the *ATIS Options Set*. Following identification of these properties, the *Associated Axioms* are then selected. *Associated Axioms* are those in which one or more of the selected properties occur. With the selection of these axioms, an analysis of the system is possible using the Predicate, Relation, and Set Calculi herein developed.

It is also intended that a topological analysis will eventually be possible either by the direct use of operations taken from mathematical topology or a derivation thereof. Such an analysis, along with other analytic techniques, is beyond the scope of this report.

The following axiom asserts that if we have a set of specific predicate-defined components and a set defined by relations of those components, then we have a *System*, $\mathbb{S}$. Axiom Schema (17) is the *System* axiom schema.

Axiom (17) $\hat{w}F(w) \equiv \mathbb{S}_{\mathbf{x}} \wedge \hat{y}F(y) \equiv \mathbb{S}_{\phi} :\equiv: \mathbb{S} = (\mathbb{S}_{\mathbf{x}}, \mathbb{S}_{\phi})$

If $\mathbb{S}_{\mathbf{x}}$ is a partitioned set, $\mathcal{P}$, and $\mathbb{S}_{\phi}$ is a family of affect relations, $\mathcal{A}$, that determine a list of properties, $\mathcal{D}$, then we can assign transition functions, $\mathcal{T}$, a time function, $\mathcal{T}$, and a system state transition function, σ.

Axiom (18) asserts that for every property, $\mathcal{D}$, there exists a property qualifier that determines the class $\hat{w}\mathcal{D}(w)$.

Axiom (18) $\forall \mathcal{D}(w) \exists \hat{w}(\hat{w}\mathcal{D}(w))$

The following axiom asserts that if we have a property class by Axiom (18), then there is a morphism that can be defined between that class and another property class.

Axiom (19) $\hat{w}\mathcal{D}(w) \supset \exists \underline{X} \exists \hat{y}\mathcal{D}(y)(\underline{X}(\hat{w}\mathcal{D}(w) \rightarrow \hat{y}\mathcal{D}(y)))$

Axiom (20) asserts that if we have a System so defined as characterized above, then we have a General System.[9]

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[9] A formal definition of General System is provided in a companion paper, Report #2-1, entitled *"General System" Defined for Predictive Technologies of A-GSBT* (Axiomatic-General Systems Behavioral Theory), which has been accepted for publication to *Scientific Inquiry Journal*, a web-based, peer-reviewed journal.

Axiom (20) $\mathbb{S} \mid \exists \hat{g} \mathcal{P} \subset \mathbb{S}_\mathbf{x} \exists \hat{h} \mathcal{A} \subset \prod \mathbb{S}_\mathbf{x} \exists \hat{i} \mathcal{T} \exists \mathcal{T} \exists \sigma :\supset: \exists \mathcal{G}(\mathcal{G} = (\mathcal{P}, \mathcal{A}, \mathcal{T}, \mathcal{T}, \sigma))$

Given a General System, $\mathcal{G}$, then the ATIS Options Set, $\mathcal{O}$, is defined by the set of system properties, $\mathcal{D}$, and their associated axioms, $\mathcal{A}$

$$\mathcal{O}_{\text{ATIS}} =_{\text{df}} \hat{w}_{i=1:n} \mathcal{D}_i(w_i) \cup \hat{y}_{j=1:n} \mathcal{A}_i(y_j)_{\mathcal{D}_i}$$

It is this *ATIS Options Set* that determines the properties and axioms that are used to analyze a system. The observed affect relations of the system first identify the properties. With the identification of the properties, the axioms are selected as those in which the properties are exhibited as one of the parameters. Once the axioms are identified, the axiom schemas are used to determine an initial analysis of the system. Further analyses can be obtained by use of the SCTs or other analytic techniques.